Exercise 2 Stochastic Models of Manufacturing Systems 4T400, 28 April

1. Consider a bin with 10 balls: 5 are red, 3 are green and 2 are blue. You randomly pick 3 balls from the bin. What is the probability that each of these balls has a different color?

   Answer: There are $\binom{10}{3}$ equiprobable combinations of 3 balls from a bin of 10 balls and $5 \times 3 \times 2$ of these combinations are such that each ball has a different color. Hence, the probability is $\frac{5\times3\times2}{\binom{10}{3}} = \frac{1}{4}$.

2. Jan and Kees are in a group of 12 people that are seated randomly at a round table (with 12 seats). What is the probability that Jan and Kees are seated next to each other?

   Answer: Jan can be seated at any of the 12 seats, from the remaining 11 seats, Kees must be seated at 1 of the 2 seats next to Jan. So the probability is $\frac{2}{11}$.

3. A batch of 100 processors is shipped simultaneously. Before shipping, 20 processors are tested. A batch is shipped if at most 1 processor is broken. What is the probability that a batch with 4 broken processors is shipped?

   Answer: Probability of no broken processor is $\binom{96}{20}/\binom{100}{20} = 0.4033$, and probability of one broken processor is $\binom{4}{1}\binom{96}{19}/\binom{100}{20} = 0.4191$. So probability of at most one broken processor is $0.4033 + 0.4191 = 0.8224$.

4. On a shelf there are 19 books: 10 Science books, 5 physics books and 4 chemistry books. You randomly pick 2 books. What is the probability that the two books are of the same subject?

   Answer: Condition on the first book that you pick. Hence, probability that the second book is of the same subject as the first is $\frac{10}{19}\frac{9}{18} + \frac{5}{19}\frac{4}{18} + \frac{4}{19}\frac{3}{18} = 0.3536$.

5. A machine produces parts that are either good (90%), slightly defective (3%) and obviously defective (7%). All produced parts are automatically inspected for defects. The defect of an obviously defective part is detected and the obviously defective part is discarded. All other parts pass the inspection and are not discarded. What is the probability that a part that passed the inspection is defective?

   Answer: A part can be good (G), slightly defective (SD) or defective (D). Then

   $$P(\{SD\}|\{SD, G\}) = \frac{0.03}{0.9 + 0.03} = 0.0323.$$

6. Consider an equilateral triangle (gelijkzijdige driehoek). Denote the length of each of the edges by X. The probability that $X = 1, 2, 3$ equals $\frac{1}{3}$. What is the expected area of the equilateral triangle?

   Answer: The area of an equilateral triangle with the length of its edges being x equals $x^2\frac{\sqrt{3}}{4}$. Hence,

   $$E(X) = \frac{\sqrt{3}}{4}\left(\frac{1}{3}\cdot 1^2 + \frac{1}{3}\cdot 2^2 + \frac{1}{3}\cdot 3^2\right) = 2.0207.$$

7. Consider an equilateral triangle (gelijkzijdige driehoek). Denote the length of each of the edges by X, where X is a continuous variable with a uniform density over interval $(1, 3)$. What is the expected area of the equilateral triangle?

Answer:

$$E(X) = \int_{x=1}^{3} x^2 \frac{\sqrt{3}}{4} \frac{1}{2} dx = \frac{13\sqrt{13}}{12}.$$

8. Let X be a positive random variable with distribution function

$$P(Xx) = F_X(x) = 1 - \frac{c}{2+x}.$$

- Determine the constant c.

  Answer: $F_X(0) = 0$ yields $c = 2$, or

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{c}{(2+x)^2},$$

  and since

$$1 = \int_0^{\infty} f_X(x) dx = \frac{c}{2},$$

  we get $c = 2$.

- Calculate the conditional probability $P(X > 8 | X \leq 10)$.

  Answer:

$$P(X > 8 | X \leq 10) = \frac{P(8 < X \leq 10)}{P(X \leq 10)} = \frac{\int_8^{10} f_X(x) dx}{\int_0^{10} f_X(x) dx} = \frac{\frac{1}{30}}{\frac{5}{6}} = \frac{1}{25}.$$

9. We select a point (x, y) at random from a square with sides 1. Let the random variable X denote the x-coordinate and Y the y-coordinate of that point. What is the probability that $X > 1.5 - Y$?

Answer:

$$P(X > 1.5 - Y) = \int_{y=0.5}^{1} \int_{x=1.5-y}^{1} dx dy = \frac{1}{8}.$$